KABAT'S SURFACE TERMS IN THE ZETA-FUNCTION APPROACH

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The thermal partition functions of photons in any covariant gauge and gravitons in the harmonic gauge, propagating in a Rindler wedge, are computed using a local zeta-function approach. The results are discussed in relation to the quantum corrections to the black hole entropy. The correct leading order temperature dependence T^4 is obtained in both cases. For the photons, it is confirmed the existence of a surface term giving a negative contribution to the entropy, as earlier obtained by D.Kabat, but this term is shown to be gauge-dependent in the four dimensional case and therefore discarded. It is argued that similar terms could appear dealing with any integer spin $s \geq 1$ in the massless case and in more general manifolds. Our conjecture is checked in the case of a graviton in the harmonic gauge, where different surface terms also appear.

1 Zeta function approach for photons in the Rindler wedge

The aim of this work is the computation of the one-loop quantum corrections to the black-hole entropy due to photons and gravitons. In our approach, the corrections are identified with the entropy of the quantum fields living outside the horizon.

In the last years, many papers have dealt with the computation of the one-loop quantum correction to the entropy of a large mass black-hole, using the approximation of the Schwarzschild metric given by the simpler Rindler metric and the conical singularity method. Most of these works have considered the scalar field only,

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while, last year D.Kabat published a paper[1], in which he explicitly considered the fermion and photon fields. In the photon case, he found an unexpected 'surface' term in the effective action, in the sense that it can be interpreted as due to particle paths beginning and ending at the black-hole surface (horizon). This term makes negative the corrections to the black-hole entropy at the the Hawking temperature.
Kabat employed the heat-kernel plus proper time regularization procedure which notoriously gives a wrong temperature dependence of the thermodynamical quantities in dimensions other than two.

For this reason, we investigated the photon case with the local zeta-function approach developed by Zerbini, Cognola and Vanzo [2] for the scalar case and which gives the correct temperature behaviour.
We have computed the thermal partition function of a quantum field employing an Euclidean path integral over all the field configurations that are periodic in the imaginary time and identify the period β with the inverse temperature: in doing this, the Rindler wedge becomes the manifold $C_\beta \times \mathrm{R}^2$,where C_β is the cone with angular deficit $2\pi - \beta$. The one-loop thermal partition function is then given by

$$\ln Z_\beta = -\frac{1}{2} \ln \det L_4 + \ln Z_\beta^{\mathrm{Ghost}}$$

where, in our case, L_4 is the small fluctuation operator of the e.m. field, that we have considered in any covariant gauge:

$$L_4 = [g^{ab}(-\Delta) + (1 - \frac{1}{\alpha})\nabla^a \nabla^b],$$

where α is the gauge fixing parameter and g^{ab} is the Euclidean Rindler metric:

$$ds^2 = r^2 d\tau^2 + dr^2 + dy^2 + dz^2, \quad 0 < \tau < \beta, \, , 0 < r < \infty, \quad y, z \in \mathrm{R}$$

To compute the above determinant, the first step has been to find a complete set of delta normalized eigenfunctions for the small fluctuation operator on the manifold with conical singularity $C_\beta \times \mathrm{R}^2$:

$$
\begin{aligned}
A_a^{(I,n\lambda\mathbf{k})} &= \frac{1}{k}\epsilon_{ij}\partial^j\phi = \frac{1}{k}(0,0,ik_z\phi,-ik_y\phi),\\
A_a^{(II,n\lambda\mathbf{k})} &= \frac{1}{\lambda}\sqrt{g}\epsilon_{\mu\nu}\nabla^\nu\phi = \frac{1}{\lambda}(r\partial_r\phi,-\frac{1}{r}\partial_\tau\phi,0,0),\\
A_a^{(III,n\lambda\mathbf{k})} &= \frac{1}{\sqrt{\lambda^2+\mathbf{k}^2}}(\frac{k}{\lambda}\nabla_\mu-\frac{\lambda}{k}\partial_i)\phi = \frac{1}{\sqrt{\lambda^2+\mathbf{k}^2}}(\frac{k}{\lambda}\partial_\tau\phi,\frac{k}{\lambda}\partial_r\phi,-\frac{\lambda}{k}\partial_y\phi,-\frac{\lambda}{k}\partial_z\phi),\\
A_a^{(IV,n\lambda\mathbf{k})} &= \frac{1}{\sqrt{\lambda^2+\mathbf{k}^2}}\nabla_a\phi = \frac{1}{\sqrt{\lambda^2+\mathbf{k}^2}}(\partial_\tau\phi,\partial_r\phi,\partial_y\phi,\partial_z\phi),
\end{aligned}
$$

where $\phi = \phi_{n\lambda\mathbf{k}}(x)$ is the complete set of delta normalized eigenfunctions of the Friedrichs self-adjoint extension of the scalar Laplacian on $C_\beta \times \mathrm{R}^2$ with eigenvalues $-(\lambda^2+\mathbf{k}^2)$:

$$
\phi_{n\lambda\mathbf{k}}(x) = \frac{\sqrt{\lambda}}{2\pi\sqrt{\beta}}e^{ik_y y+ik_z z}e^{i\frac{2\pi n}{\beta}\tau}J_{\nu_n}(\lambda r), \quad n=0,\pm1,\ldots;\ \lambda\in\mathrm{R}^+;\ k_y,k_z\in\mathrm{R}
$$

The first three eigenfunctions satisfy $\nabla^a A_a = 0$ and have eigenvalue $\lambda^2+\mathbf{k}^2$, while $A_a^{(IV)}$ is a pure gauge and has eigenvalue $\frac{1}{\alpha}(\lambda^2+\mathbf{k}^2)$. Using these modes we have then written the local zeta-function of the e.m. field using its spectral representation:

$$
\zeta(s;x) = \sum_i\sum_n\int d\lambda\, d^2\mathbf{k}\ [\nu_i^2(n\lambda\mathbf{k})]^{-s}g^{ab}A_a^{(i)}(x)A_b^{(i)*}(x)\,.
$$

Since the manifold is non-compact, only the local zeta-function is directly definable as a finite quantity; to define the global zeta-function one needs to introduce some smearing function in the spatial integrations.
After some manipulations, we have found that the zeta-function

can be expressed in terms on the zeta-function of a minimally coupled scalar field and a new term which is a total derivative:

$$\zeta(s;x) = (3+\alpha^s)\zeta_\beta^{\rm Scalar}(s;x) + \frac{s+1+\alpha^s(s-1)}{2(s-1)}\Delta\zeta_\beta^{\rm Scalar}(s+1;x).$$

where the scalar local zeta-function in this background has been computed by Zerbini, Cognola and Vanzo [2]:

$$\zeta_\beta^{\rm Scalar}(s;x) = \frac{r^{2s-4}}{4\pi\beta\Gamma(s)} I_\beta(s-1),$$

where $I_\beta(s)$ is a function analytic in the whole complex plane but in $s=1$, where it has a simple pole, and its value in the interesting points $s=0,-1$ are known [2].
After adding the contribution of the ghosts [c], which is $-2\alpha^{(s/2)}\zeta_\beta^{\rm Scalar}(s;x)$, and using the formulae

$$\begin{aligned} \mathcal{L}_\beta(x) &= \frac{1}{2}\zeta'(s=0;x) + \frac{1}{2}\zeta(s=0;x)\ln\mu^2, \\ \ln Z_\beta &= \int d^4x\sqrt{g}\,\mathcal{L}_\beta(x), \end{aligned}$$

we find that the one loop effective Lagrangian density for the electromagnetic field on $C_\beta \times {\rm R}^2$ is

$$\begin{aligned} \mathcal{L}_\beta^{\rm e.m.}(x) &= 2\mathcal{L}_\beta^{\rm Scalar}(x) - \frac{(1-\frac{1}{2}\ln\alpha)}{2\pi\beta r^4} I_\beta(0) \\ &= \frac{1}{4\pi\beta r^4} I_\beta(-1) - \frac{(1-\frac{1}{2}\ln\alpha)}{2\pi\beta r^4} I_\beta(0). \end{aligned}$$

Some comments on the found results are in order.
The twice-scalar part is expected from the physical ground. It is

[c]This contribution follows by a direct zeta-function approach on the gauge fixing Faddeev-Popov determinant (which appears in the complete photon Feynman integral) taking its $\alpha-$dependence into account.

also clear that $\mathcal{L}_\beta^{\text{e.m.}}(x)$ may depend on the gauge fixing parameter α since the gauge invariance must hold for integrated quantities like the effective action [d].
Anyway, when integrated, the gauge-dependent term gives rise to Kabat-like surface term and gives a negative contribution to the entropy, at least for some values of α. However, since the gauge-dependent part take rise form a total derivative, such a term would be discarded after the integration over the space-time if this were a smooth, compact manifold without borders or singularities: this is a standard procedure when checking the gauge invariance of a theory. As a consequence, our procedure to restore the gauge invariance is simply to discard the 'surface' term: in this way we also avoid embarrassing negative entropies. This procedure reflects the fact that local quantities like the the local zeta-function are ill-defined due to the possibility of adding a total derivative with vanishing integral.
After discarding the 'surface' term, we can compute the thermodynamical quantities in the usual way: for example the renormalized free energy, the energy density and the entropy are:

$$\begin{aligned} F_\beta^{\text{Sub.}} &= -\frac{A_\perp \pi^2}{90\beta^2\epsilon^2} - \frac{A_\perp}{36\beta^2\epsilon^2} - \frac{13A_\perp}{1440\pi^2\epsilon^2}, \\ < T_0^0 >^{\text{Sub.}} &= \frac{\pi^2}{15\beta^4 r^4} + \frac{1}{18\beta^2 r^4} - \frac{13}{720\pi^2 r^4}, \\ S_\beta &= \beta^2 \partial_\beta F_\beta = \frac{A_\perp}{90\beta\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^2 + 5\right]. \end{aligned}$$

These results are in agreement with twice the minimally coupled scalar field results obtained with different methods (point-splitting, optical metric) [3,4,5] but for the coefficient of the term β^{-2} for which

[d] One can notice that the two-dimension case is misleading, since the one-loop effective lagrangian density is indeed gauge-invariant.

we get one third of their result. This discrepancy appears also in the heat-kernel approach [6] and is not yet understood.

2 A general conjecture and the graviton case

Investigating the origin of the surface gauge-dependent term we have conjectured the appearence of such terms on more general manifolds in the form $\mathcal{M} \times \mathrm{R}^2$, where the generally curved manifold $\mathcal{M}$ has conical singularities or boundaries, and for higher spin fields. The general form of such terms as far as the total photon zeta function contribution is concerned should be there:

$$\zeta^{\text{surface}}_{\mathcal{M}\times\mathrm{R}^2}(s;x) = \frac{s+1+\alpha^s(s-1)}{2s}\,\frac{\Gamma(s-1)}{4\pi\Gamma(s)}\nabla_a \sum_n \int d\lambda\lambda\,\mathbf{J}^*\nabla^a\mathbf{J}\,.$$

where $\mathbf{J}_{n,\lambda}(x^\mu)$ is an eigenfunction of (the Friedrichs extension of) the 0-forms Hodge Laplacian on $\mathcal{M}$.
We have checked our hypothesis in the case of the graviton field in the Rindler wedge employing the harmonic gauge, and we have indeed found many and more complicated surface-like terms, dropping which one gets the expected physical result for the effective action:

$$\ln Z^{\text{Gravitons}}_\beta = 2\ln Z^{\text{Scalar}}_\beta(s;x).$$

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